Orientations of digraphs almost preserving diameter

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Abstract

An orientation of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for every pair $x \neq y$ of vertices such that both xy and yx are in D. In this paper, we consider certain well-known classes of strong digraphs, each member D of which has an orientation with diameter not exceeding the diameter of D by more than a small constant.

1 Introduction, terminology and notation

An *orientation* of a digraph D is a spanning subdigraph of D obtained from D by deleting exactly one arc between x and y for every pair $x \neq y$ of vertices such that both xy and yx are in D. In this paper, we consider certain well-known classes of generalizations of tournaments, each strongly connected member D of which has an orientation with diameter not exceeding the diameter of D by more than a small constant. While there is a large number of publications considering minimum diameter orientations of undirected graphs (see Sections 2.6-2.9 in [2] for results and references), the present paper is the first study of minimum diameter orientations of digraphs. It is shown in Section 2.11 of [2] that orientations H of digraphs D such that the diameter of H does not exceed the diameter of D by more than a small constant are of interest in a version of the gossip problem, see, e.g., [9, 10].

It is worth noting that there are a few papers [6, 8, 11] considering finite diameter orientations of mixed graphs (or, equivalently, of directed graphs), but none of these papers has been devoted to minimizing the diameter of an orientation of a given digraph. We restrict our attention to special classes of digraphs since even the problem of checking whether a given undirected graph has an orientation of diameter 2 is proved to be $\mathcal{NP}$-complete by Chvátal and Thomassen [7] and the upper bound on the diameter of an orientation of an undirected graph obtained in [7] is far from best possible for many classes of undirected graphs. Notice that the minimum diameter orientation problem for undirected graphs is a special case of that for directed graphs since every undirected graph can be considered as the corresponding symmetric digraph.

This paper is organized as follows. In the rest of this section we give some terminology and notation. In Section 2 we prove a somewhat surprising upper bound for the minimum diameter of orientations of quasi-transitive digraphs and semicomplete bipartite digraphs. In particular, we show that if D is a strong quasi-transitive digraph on at least 3 vertices, then D has an orientation H such that $\mathrm{diam}(H) \le \max\{3, \mathrm{diam}(D)\}$. The same bound, with 3 replaced by 5, holds for all semicomplete bipartite digraphs except for those in which one partite set consists of a unique vertex. While such a bound is not valid for the whole class of locally semicomplete digraphs, in Section 3 we prove that the bound $\mathrm{diam}(H) \le \max\{4, \mathrm{diam}(D)\}$ holds for locally semicomplete digraphs D without so-called similar vertices and $\mathrm{diam}(H) \le \max\{4, \mathrm{diam}(D)\}+1$ is true for every locally semicomplete digraph D on at least three vertices.

We use the standard terminology and notation on digraphs as described in [2]. We still provide most of the necessary definitions for the convenience of the reader. A digraph D is *symmetric* if for every pair $x \neq y$ of vertices in D either there is no arc between x and y or both xy and yx are in D. Symmetric digraphs are in natural correspondence to undirected graphs: for an undirected graph G, the symmetric digraph $\overset{\leftrightarrow}{G}$ is obtained from G by replacing every edge xy with the pair xy, yx of arcs. Let $D = (V, A)$ be a digraph and let x, y be a pair of vertices in D. If $xy \in A$, we say that y is an *out-neighbour* of x, x is an *in-neighbour* of y, and x *dominates* y denoted by $x{\rightarrow}y$. For sets $X, Y \subset V$, $X{\rightarrow}Y$ means that $x{\rightarrow}y$ for every $x \in X$, $y \in Y$. The set of in-neighbours (out-neighbours) of a vertex x is denoted by $N^-(x)$ ($N^+(x)$).

All paths and cycles we consider in this paper are directed. A path from x to y is an *(x, y)-path.* A digraph D is *strongly connected* (or, *strong*) if there exist an (x, y)-path and a (y, x)-path for every pair x, y of distinct vertices in D. The *distance,* $\mathrm{dist}_D(x, y)$, *from x to y* in D is the least length of an (x, y)-path if y is reachable from x, and is equal to ∞, otherwise. We assume that $\mathrm{dist}_D(x, x) = 0$ for every vertex $x \in V$. The *diameter* of D, $\mathrm{diam}(D)$, is the maximum distance between an ordered pair of vertices in D. Observe that a digraph D is strong if and only if $\mathrm{diam}(D) < \infty$. A digraph D is *connected* if the underlying undirected graph of D is connected. For a digraph D, let $\mathrm{diam}_{\min}(D)$ denote the minimum diameter of an orientation of D. The *converse* of a digraph D is the digraph obtained from D by replacing every arc xy of D by the arc yx.

A digraph D is *semicomplete* if there is at least one arc between any pair of distinct vertices of D. A *tournament* is a semicomplete digraph with no cycle of length 2. A digraph D is *quasi-transitive* if the existence of a pair xy, yz of arcs in D implies the existence of

xz or zx (or both). By definition, every semicomplete digraph is quasi-transitive. To see that there are quasi-transitive digraphs, which are not semicomplete (and not transitive), replace every vertex of a tournament T by a set of independent (i.e. with no arc between them) vertices. The resulting digraph D is quasi-transitive: if xy, yz are in D, then x and y belong to different sets of independent vertices (as T has no 2-cycle) and, thus, are joint by an arc. A recursive characterization of quasi-transitive digraphs is given by Bang-Jensen and Huang [5].

A digraph D is *locally semicomplete* if, for every vertex x, the subdigraphs of D induced by $N^+(x)$ and $N^-(x)$ are semicomplete. One of the simplest examples of a locally semicomplete digraph is a cycle. A digraph D is semicomplete k-partite, $k \geq 2$, if the vertices of D can be partition into k *partite sets* $V_1, V_2, ..., V_k$ such that every partite set is independent, but, for every pair x, y of vertices from distinct partite sets, xy or yx (or both) is in D. When $k = 2$, we speak of semicomplete bipartite digraphs. By definition, every semicomplete digraph with n vertices is a semicomplete n-partite digraph. A characterization of locally semicomplete digraphs is obtained in [1].

Quasi-transitive digraphs, locally semicomplete digraphs and semicomplete k-partite digraphs are well-known generalizations of tournaments, they share several nice structural properties with tournaments and have been extensively studied in the literature (cf. [2, 3] and the bibliography therein). In particular, we know now that the hamiltonian cycle is polynomial time solvable when restricted to any of these classes. (A highly non-trivial proof that the hamiltonian cycle problem is polynomial time solvable for semicomplete k-partite digraphs can be found in [4].)

We conclude this section with the following useful result by Boesch and Tindell [6], whose short proof is given by Volkmann [11]:

Theorem 1.1 *A strong digraph D has no strong orientation if and only if there is a pair x, y of vertices in D such that the deletion of the arcs xy, yx leaves D disconnected.*

2 Orientations of quasi-transitive digraphs and semicomplete bipartite digraphs

Applying Theorem 1.1 it is easy to see that every strong quasi-transitive digraph of order $n \geq 3$ has a strong orientation. Volkmann [11] observed that a strong semicomplete k-partite digraph D, $k \geq 2$, has a strong orientation unless D is a semicomplete bipartite digraph with a partite set consisting of a single vertex. (By Theorem 1.1, a semicomplete bipartite digraph with a partite set consisting of a single vertex does not have a strong orientation.) This justifies the consideration of the following two classes of digraphs. Let $\mathcal{D}_0$ be the set of strong quasi-transitive digraphs of order $n \geq 3$. Let $\mathcal{D}_1$ be the set of strong semicomplete bipartite digraphs with at least two vertices in each partite set.

In this section, we shall use the following basic result:

Proposition 2.1 *[5] Let D be a quasi-transitive digraph. Suppose that $P = x_0x_1x_2...x_k$ is a minimal (x_0, x_k)-path. Then the subdigraph induced by $V(P)$ is semicomplete and $x_j \rightarrow x_i$ for every $2 \leq i+1 < j \leq k$, unless $k = 3$, in which case the arc between x_0 and x_k may be absent.*

For digraphs from the class $\mathcal{D}_0 \cup \mathcal{D}_1$ the following somewhat surprising bound on the minimum diameter of an orientation holds.

Theorem 2.2 *If $D \in \mathcal{D}_i$ for $i \in \{0,1\}$, then*

$$\mathrm{diam}_{\min}(D) \leq max\{3+2i, \mathrm{diam}(D)\}.$$

Proof: Assume that this theorem is false and that D is a counter-example to the theorem with as few 2-cycles as possible. Let $D \in \mathcal{D}_i$ for $i \in \{0,1\}$ and let $\gamma = 3+2i$. Let xyx be a 2-cycle in D. Clearly, the diameter of D increases by at least one when we delete either of the arcs xy or yx from D. Therefore, there exist vertices $s_{xy}, t_{xy}, s_{yx}, t_{yx}$ in D, such that $\mathrm{dist}_{D-xy}(s_{xy}, t_{xy}) > \max\{\gamma, \mathrm{diam}(D)\}$ and $\mathrm{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{\gamma, \mathrm{diam}(D)\}$. Let $P = p_0p_1 \ldots p_l$ be an (s_{xy}, t_{xy})-path in D of minimum length (in particular, $l \leq \mathrm{diam}(D)$) and let $Q = q_0q_1 \ldots q_m$ be an (s_{yx}, t_{yx})-path in D of minimum length (in particular, $m \leq \mathrm{diam}(D)$). Let ρ and η be defined such that $xy = p_\rho p_{\rho+1}$ and $yx = q_\eta q_{\eta+1}$.

We now consider the following cases, which exhaust all possibilities:

Case 1: $\rho+1 < l$, $\eta+1 < m$ and $D \in \mathcal{D}_0 \cup \mathcal{D}_1$. We first show that $p_{\rho+2}$ and $q_{\eta+2}$ are adjacent. This is clearly true if D is semicomplete bipartite as these two vertices belong to different partite sets of D. If D is quasi-transitive, then p_ρ and $p_{\rho+2}$ are adjacent. Therefore, $p_{\rho+2} \rightarrow p_\rho$ by the minimality of l. However, this implies that $p_{\rho+2}$ and $q_{\eta+2}$ are adjacent, as $p_{\rho+2} \rightarrow (p_\rho = q_{\eta+1}) \rightarrow q_{\eta+2}$.

If $p_{\rho+2} \rightarrow q_{\eta+2}$, then by $q_\eta = p_{\rho+1}$,

$$q_0q_1 \cdots q_\eta p_{\rho+2} q_{\eta+2} \cdots q_m$$

is a (q_0, q_m)-path of length $m \leq \mathrm{diam}(D)$ in $D - yx$, a contradiction. The case when $q_{\eta+2} \rightarrow p_{\rho+2}$ can be considered analogously.

Case 2: $\rho > 0$, $\eta > 0$ and $D \in \mathcal{D}_0 \cup \mathcal{D}_1$. This case can be transformed into Case 1 by considering the converse of D.

Case 3: $\rho = 0$, $\eta + 1 = m$ and $D \in \mathcal{D}_0$. We first prove that $l + m \geq 3$. Suppose that $l = m = 1$, i.e. $x = p_0 = q_1, y = p_1 = q_0$. Let $z_0z_1...z_k$ be a shortest (y, x)-path in $D - yx$.

By the choice of x, y, $k \geq 4$. By Proposition 2.1, $z_k \to z_1$ and $z_2 \to z_0$. Hence, $z_k z_1 z_2 z_0$ is an (x, y)-path in $D - xy$ of length three, contradiction. Therefore, we may assume, without loss of generality, that $l \geq 2$.

Let $R = r_0 r_1 \dots r_t$ be a shortest path from q_0 to p_l in D. The path R can be chosen such that it does not contain yx. Indeed, if $y = r_j, x = r_{j+1}$ for some j, then $r_0 r_1 ... r_j p_2 p_3 ... p_l$ is not longer than R (as $p_1 p_2 ... p_l$ is a shortest (p_1, p_l)-path in D). So, we may assume that R does not contain yx. Similarly, it is not difficult to see that we may assume that R does not contain xy.

By Proposition 2.1, we obtain immediately that $p_l \to p_0$ if $l \neq 3$ and $p_l \to p_1$ if $l = 3$. If $l = 3$, then we have $p_3 \to p_1$ and $p_0 \to p_1$. Therefore, by the minimality of l, $p_3 \to p_0$. Hence, for every $l \geq 2$, $p_l \to p_0$.

We have $t > 2$, for otherwise $r_0 r_1 \dots r_t p_0$ would be a path from q_0 to q_m of length $t + 1 \leq 3$ in $D - yx$. Since $p_l \to p_0$ and $r_{t-1} \to r_t = p_l$, we conclude that r_{t-1} and p_0 are adjacent. If $r_{t-1} \to p_0$, then $r_0 r_1 \dots r_{t-1} p_0$ is a path from q_0 to q_m of length $t \leq \mathrm{diam}(D)$ in $D - yx$, a contradiction. If $p_0 \to r_{t-1}$, then $p_0 r_{t-1} p_l$ is a path of length two from p_0 to p_l in $D - xy$, a contradiction.

Case 4: $\eta = 0$, $\rho + 1 = l$ and $D \in \mathcal{D}_0$. This case can be transformed into Case 3 by considering the converse of D.

Case 5: $\rho = 0$, $\eta + 1 = m$ and $D \in \mathcal{D}_1$. Suppose that $l = m = 1$. Let $z_0 z_1 ... z_k$ be a shortest (y, x)-path in $D - yx$. By the choice of x, y, $k \geq 6$. By the minimality of k, $z_3 \to z_0$ (z_0 and z_3 belong to different partite sets of D) and $z_k \to z_j$, where $j = 2$ or 3 (z_k and z_j belong to different partite sets of D). Hence, either $z_k z_3 z_0$ or $z_k z_2 z_3 z_0$ is an (x, y)-path in $D - xy$, a contradiction. So, we may assume, without loss of generality, that $m \geq 2$.

Let $R = r_0 r_1 \dots r_t$ be a shortest path from q_0 to p_l in D. As in Case 3, we may assume that R contains neither xy nor yx.

Suppose that $t = 0$, implying that $q_0 = p_l$ and $l, m \geq 2$. Assume that $l \geq 3$. If p_0 and p_l belong to different partite sets of D, then, by the minimality of l and the assumption that D is semicomplete bipartite, $p_l \to p_0$, which is impossible as $p_l p_0$ is a (q_0, q_m)-path of length one in $D - yx$, a contradiction. If p_0 and p_l belong to the same partite set of D, then $p_l \to p_1$ (by the minimality of l) and $p_l p_1 p_2 p_3 p_0$ is a (q_0, q_m)-path of length four in $D - yx$, a contradiction. So, $l = 2$. Analogously, we can prove that $m = 2$. Since $D - xy$ has a (p_0, p_2)-path and $p_2 = q_0 \to q_1 = p_1$, there is a (p_0, p_1)-path $S = s_0 s_1 \dots s_a$ in $D - xy$. Assume that S has minimum length and observe that $a \geq 5$, as $s_0 s_1 \dots s_a p_l$ is a (p_0, p_l)-path in $D - xy$. Furthermore, $s_3 \to s_0$ as s_0 and s_3 lie in different partite sets of D and S is of minimum length. Observe that if $p_2 \to s_3$, then $p_2 s_3 s_0$ is a (q_0, q_m)-path in $D - yx$ of length 2, and if $s_3 \to p_2$ then $s_0 s_1 s_2 s_3 p_2$ is a (p_0, p_l)-path in $D - xy$ of length 4. In both cases we obtain a contradiction. Hence, $t > 0$.

Suppose that $1 \leq t \leq 2$. Clearly r_0 and r_1 lie in different partite sets, so we may assume, without loss of generality, that r_0 and p_0 are adjacent (the case when r_1 and p_0 are adjacent can be considered analogously). Clearly p_0 dominates r_0 by the minimality of m. However, $p_0r_0 \ldots r_t$ is a (p_0, p_l)-path in $D - xy$ of length of $t + 1 \leq 3$, a contradiction. Hence, $t \geq 3$.

Clearly r_1 and r_2 lie in different partite sets, so we may assume, without loss of generality, that r_1 and p_0 are adjacent (the case when r_2 and p_0 are adjacent can be considered analogously). Clearly p_0 dominates r_1 by the minimality of m. However the path $p_0r_1 \ldots r_t$ in $D - xy$ is of length $t \leq \mathrm{diam}(D)$.

Case 6: $i_\eta = 0$, $i_\rho + 1 = l$ and $D \in \mathcal{D}_1$. This case can be transformed into Case 5 by considering the converse of D. $\square$

The upper bound of this theorem is sharp as one can see from the following examples. Let T_k, $k \geq 3$, be a (transitive) tournament with vertices $x_1, x_2, ..., x_k$ and arcs x_ix_j for every $1 \leq i < j \leq k$. Let y be a vertex not in T_k, which dominates all vertices of T_k but x_k and is dominated by all vertices of T_k but x_1. The resulting semicomplete digraph D_{k+1} has diameter 2. However, the deletion of any arc of D_{k+1} between y and the set $\{x_2, x_3, ..., x_{k-1}\}$ leaves a digraph with diameter 3. Indeed, if we delete yx_i, $2 \leq 2 \leq k-1$, then a shortest (x_k, x_i)-path becomes of length 3.

Let H be a strong semicomplete bipartite digraph with the following partite sets V_1 and V_2 and arc set A: $V_1 = \{x_1, x_2, x_3\}$, $V_2 = \{y_1, y_2, y_3\}$, and

$$A = \{x_1y_1, y_1x_1, x_1y_2, y_3x_1, x_2y_1, y_2x_2, y_3x_2, y_1x_3, x_3y_3, x_3y_2\}.$$

Let $H' = H - x_1y_1$ and $H'' = H - y_1x_1$. It is easy to verify that $\mathrm{diam}(H) = 4$ (in particular, $\mathrm{dist}(y_2, y_3) = 4$) and that $\mathrm{diam}(H') = \mathrm{diam}(H'') = 5$ (a shortest (x_1, y_3)-path in H' and a shortest (y_2, x_1)-path in H'' are of length 5). The digraph H can be used to generate an infinite family of semicomplete bipartite digraphs with the above property: replace, say, x_3 by a set of independent vertices.

3 Orientations of locally semicomplete digraphs

Unfortunately, the bound of the type

$$\mathrm{diam}_{min}(D) \leq max\{c, \mathrm{diam}(D)\}, \tag{1}$$

where c is a constant, is not valid for the whole class of strong locally semicomplete digraphs. Consider the following digraph $D_k = (V, A)$:

$$V = \{x_1, x_2, ..., x_k\}, \ A = \{x_ix_{i+1} : \ i = 1, 2, ..., k-1\} \cup \{x_kx_1, x_kx_2, x_1x_3, x_2x_1\}.$$

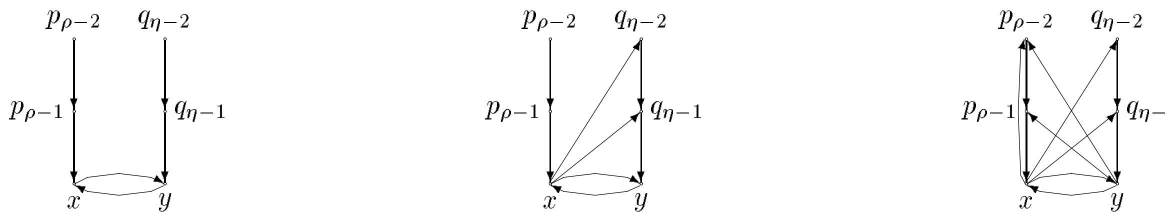

Figure 1: The leftmost picture contains the given arcs. These arcs imply $x \rightarrow q_{\eta-1}$, and thus $x \rightarrow q_{\eta-2}$, as seen in the middle picture. Analogously we obtain $y \rightarrow \{p_{\rho-2}, p_{\rho-1}\}$, which implies that $x \rightarrow p_{\rho-2}$, as seen in the last picture.

It is easy to check that $\mathrm{diam}(D_k) = k-2$ and $\mathrm{diam}(D_k - x_1x_2) = \mathrm{diam}(D_k - x_2x_1) = k-1$. The digraph D_k does not satisfy (1) due to the existence of so-called similar vertices x_1 and x_2. Two vertices x and y of a digraph D are *similar* if $N^+(x) \cup \{x\} = N^+(y) \cup \{y\}$ and $N^-(x) \cup \{x\} = N^-(y) \cup \{y\}$. Observe that if x and y are similar, then the 2-cycle xyx is in D.

The main result of this section, Theorem 3.2, can be proved using the classification of locally semicomplete digraphs obtained in [1] and Theorem 2.2 for the case of quasi-transitive digraphs (actually, for just semicomplete digraphs). Even though such a 'classification-based' proof is slightly shorter than the one we provide below, the 'classification-based' proof relies heavily on the classification and related results in [1]. The presented proof is direct and does not require any previous knowledge. Provided with enough detail, the 'classification-based' proof along with the classification itself and additional results and definitions would require more space than our proof below. We start from the following result.

Theorem 3.1 *If D is a strong locally semicomplete digraph with no similar vertices then* $\mathrm{diam}_{min}(D) \leq max\{4, \mathrm{diam}(D)\}$.

Proof: Assume that this theorem is false and that D is a counter-example, with as few 2-cycles as possible. Let xyx be a 2-cycle in D. Since x and y are not similar, we may without loss of generality find a vertex u, such that $xu \in A(D)$, but $yu \notin A(D)$. However this implies that $uy \in A(D)$, as $x \rightarrow \{u, y\}$. Since $\mathrm{diam}(D - xy) > max\{4, \mathrm{diam}(D)\}$, there are vertices s_{xy} and t_{xy} such that $\mathrm{dist}_{D-xy}(s_{xy}, t_{xy}) > \max\{4, \mathrm{diam}(D)\}$. Let $P = p_0p_1 \ldots p_l$ be a shortest (s_{xy}, t_{xy})-path in D. Since $\mathrm{dist}_{D-xy}(s_{xy}, t_{xy}) > \mathrm{diam}(D)$ the arc xy must be used in the path P, so let ρ be defined such that $xy = p_\rho p_{\rho+1}$. The path $P' = p_0p_1 \ldots p_\rho u p_{\rho+1} \ldots p_l$ is a path in $D - xy$, implying that $l = \mathrm{diam}(D) \geq 4$. If $\rho \geq 1$

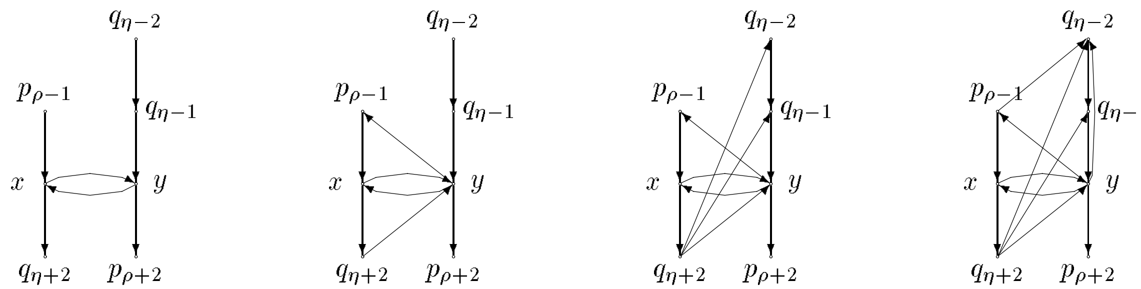

Figure 2: The leftmost picture contains the given arcs. This implies that the arcs $q_{\eta+2}{\rightarrow}y$ and $y{\rightarrow}p_{\rho-1}$ must be present, as seen in the next picture. This implies that $q_{\eta+2}{\rightarrow}q_{\eta-1}$, which implies that $q_{\eta+2}{\rightarrow}q_{\eta-2}$, as seen in picture 3. Finally we must therefore have arc $y{\rightarrow}q_{\eta-2}$, which implies that $p_{\rho-1}{\rightarrow}q_{\eta-2}$, as seen in the last picture.

then we observe that $p_{\rho+1}p_{\rho-1} \in A(D)$ (as $\{p_{\rho+1}, p_{\rho-1}\}{\rightarrow}p_\rho$ and l is minimum). If $\rho = 0$ then $p_2{\rightarrow}p_0$ by a similar argument. So there is a (y, x)-path of length 2 in $D - yx$.

There exist vertices s_{yx} and t_{yx} in D, such that $\mathrm{dist}_{D-yx}(s_{yx}, t_{yx}) > \max\{4, \mathrm{diam}(D)\}$. Analogously to the above we let $Q = q_0q_1 \dots q_m$ be a shortest (s_{xy}, t_{xy})-path in D, and observe that $yx \in A(Q)$, which implies that there is some η, such that $yx = q_\eta q_{\eta+1}$. Furthermore $m = \mathrm{diam}(D) \geq 4$, as there is a path from y to x of length 2 in $D - yx$.

Assume without loss of generality that $\eta \geq 2$, as otherwise we can reverse all arcs and swap the names x and y, in order to get $\eta \geq 2$ (this is true since $m \geq 4$). We now consider the following cases, which exhaust all possibilities:

Case 1: $\rho > 1$. Using the minimality of l and m we observe that the arguments in Figure 1 imply that $q_{\eta-2}$ and $p_{\rho-2}$ are adjacent, as $x{\rightarrow}\{q_{\eta-2}, p_{\rho-2}\}$ in the last picture. If $q_{\eta-2}{\rightarrow}p_{\rho-2}$ then the path $q_0q_1 \dots q_{\eta-2}p_{\rho-2}p_{\rho-1}q_{\eta+1} \dots q_m$ is a path of length m in $D - yx$, a contradiction. If $p_{\rho-2}{\rightarrow}q_{\eta-2}$, then we analogously arrive to a contradiction.

Case 2: $\rho = 1$ **and** $\eta + 2 \leq m$**.** Then, by the minimality of l and m, we obtain the arcs seen in the last picture of Figure 2. Since $\{p_{\rho-1}, q_{\eta+2}\}{\rightarrow}q_{\eta-2}$, the vertices $p_{\rho-1}$ and $q_{\eta+2}$ are adjacent. We cannot have $p_{\rho-1}{\rightarrow}q_{\eta+2}$ as then the path $(p_0 = p_{\rho-1})q_{\eta+2}p_{\rho+1} \dots p_l$ is a (p_0, p_l)-path of length l in $D - xy$. Therefore $q_{\eta+2}{\rightarrow}p_{\rho-1}$. However this implies that $p_{\rho-1}$ and $q_{\eta-1}$ are adjacent. We can now get a contradiction analogously to Case 1.

Case 3: $\rho = 0$. We see from Figure 3 that $x{\rightarrow}\{q_0, q_1, \dots, q_{\eta-1}\}$. Let $R = r_0r_1 \dots r_t$ be a shortest path from q_0 to p_l in D (see Figure 3). We have $t \geq 3$ as $(p_0 = x){\rightarrow}q_0$ and there is no (p_0, p_l)-path of length at most four in $D - xy$. Observe that if x and

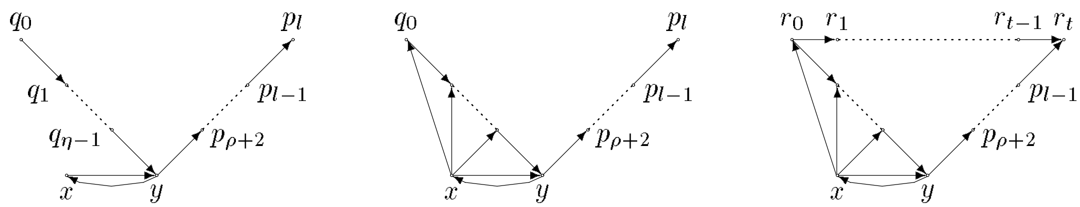

Figure 3: The first picture contains the given arcs. This implies that the arc $x \to q_{\eta-1}$, which implies that $x \to q_{\eta-2}$. Continuing this process we see that $x \to \{q_0, q_1, \ldots, q_{\eta-1}\}$, as seen in the middle picture. In the last picture we have added a shortest (q_0, p_l)-path.

r_1 are adjacent then either $q_0 r_1 q_{\eta+1} \ldots q_m$ or $p_0 r_1 r_2 \ldots r_{t-1} p_l$ are paths of length at most $\mathrm{diam}(D)$ in $D - \{xy, yx\}$ a contradiction. Therefore x and r_1 are not adjacent in D.

Since $q_0 \to \{q_1, r_1\}$ we observe that $q_1 \to r_1$, as if $r_1 \to q_1$ then x and r_1 would be adjacent (as $q_{\eta+1} \to q_1$). Analogously $q_2 \to r_1$, as $q_1 \to \{q_2, r_1\}$. Continuing in this fashion we get that $\{q_0, q_1, \ldots, q_{\eta+1}\} \to r_1$, which is a contradiction against $q_{\eta+1}$ and r_1 not being adjacent.

Case 4: $\rho = 1$ and $\eta + 2 > m$. This clearly implies that $\eta + 1 = m$, as $m \geq \eta + 1$. By reversing all arcs we obtain the case when $\rho = 0$ and $\eta = l - 2 \geq 2$, which we handled in Case 3. □

Theorem 3.2 *If D is a strong locally semicomplete digraph of order $n \geq 3$, then*

$$\mathrm{diam}_{min}(D) \leq max\{5, \mathrm{diam}(D) + 1\}.$$

Proof: For a given vertex $x \in V(D)$, let $(N^+(x) \cup \{x\}, N^-(x) \cup \{x\})$ be the neighbourhoods pair of x. Let $V_1 = (N_1, M_1), V_2 = (N_2, M_2), \ldots, V_k = (N_k, M_k)$ be the distinct neighbourhood pairs in D, and let v_i be some vertex in D with $NT(v_i) = (N_i, M_i)$, for $i = 1, 2, \ldots, k$. Let D' be the subdigraph of D induced by $\{v_1, v_2, \ldots, v_k\}$. If $k = 1$, then $D = \overleftrightarrow{K}_n$. In this case our result follows from Theorem 2.2. So, we may assume that $k \geq 2$.

We will now show that D' is a strong locally semicomplete digraph. Since D' is an induced subgraph of D, it is clearly a locally semicomplete digraph. Let v_j, v_t be a pair of distinct vertices in D' and let $P = v_j p_0 p_1 \ldots p_l v_t$ be a shortest (v_j, v_t)-path in D. Assume that $p_i \in V_{a_i}$ for all $i = 0, 1, \ldots, l$. Since P is shortest, all sets $V_j, V_{a_1}, V_{a_2}, \ldots, V_{a_l}, V_t$ are distinct. However this implies that $v_j v_{a_1} v_{a_2} \ldots v_{a_l} v_t$ is a path in D'. So D' is strong.

By Theorem 3.1 we can find an orientation D'' of D' such that

$$\mathrm{diam}(D'') \leq max\{4, \mathrm{diam}(D')\}.$$

We now let D''' be the digraph obtained from D'' by replacing every vertex v_i with the set V_i and choosing arbitrary orientations for arcs between vertices in the same V_i ($i = 1, 2, \ldots, k$). As above we can easily see that the distance between vertices in distinct sets, V_j and V_t, remains the same in D''' as in D''. Let $u \neq w \in V_i$. Since D''' is strong, there is a vertex $v \notin V_i$ such that $v \rightarrow w$. Clearly, $\mathrm{dist}_{D'''}(u, w) \leq \mathrm{dist}_{D''}(u, v) + 1$. Thus, the distance between two vertices in the same set V_i in D''', is at most $\mathrm{diam}(D'') + 1$ and D''' is an orientation of D with $\mathrm{diam}(D''') \leq \mathrm{diam}(D'') + 1 \leq max\{4, \mathrm{diam}(D')\} + 1 \leq max\{4, \mathrm{diam}(D)\} + 1$. $\square$

4 Further research

We were not able to prove or disprove the following bound for strong semicomplete k-partite digraphs D: $\mathrm{diam}_{\min}(D) \leq \mathrm{diam}(D) + c$, where c is a constant.

Since every undirected graph can be considered as the corresponding symmetric digraph, it would be interesting to see what results on diameters of orientations of undirected graphs can be extended to digraphs. The results on minimum diameter orientations of undirected graphs form only a small part in the important area of orientations of undirected graphs (e.g., Chapter 8 in [2] is completely devoted to orientations of graphs). It would be interesting to investigate what results in the area can be (or cannot be) generalized to orientations of digraphs, see Section 7.14 in [2] for some examples of such results.

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